Student Name: SOLUTIONS

Practice Exam 2

MATHEMATICS: SPECIALIST

Question/Answer Booklet – Section 2 – Calculator-assumed

Time allowed for this paper

Section	Reading	Working
~~**Calculator-free**~~	~~5 minutes~~	~~50 minutes~~
Calculator-assumed	10 minutes	100 minutes

Materials required/recommended for this paper

Section Two (Calculator-assumed): 98 marks

To be provided by the supervisor
Section Two Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the School Curriculum and Standards Authority for this course.

Important Note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. **All** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare answer pages may be found at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued (i.e. give the page number).

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Structure of this section

Questions	Marks available	Your score
8	8	
9	12	
10	5	
11	7	
12	10	
13	3	
14	11	
15	12	
16	10	
17	7	
18	13	
Total for Section 2	**98**	

Section Two: Calculator-assumed **(100 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the space provided.

Suggested working time for this section is 100 minutes.

Question 8 **(8 marks)**

Let μ and σ respectively be the mean and standard deviation for waiting time for calls to Telstra customer services.

(a) The waiting times for a random sample of 150 customers were recorded. The mean waiting time was 22.6 minutes with a sample standard deviation of 2.7 minutes. Find a 99% confidence interval for μ.

(2 marks)

C-Level	0.99
σ	2.7
x̄	22.6
n	150

Lower	22.032148
Upper	23.167852
x̄	22.6
n	150

$22.03 \leq \mu \leq 23.16$ ✓✓

(b) The waiting times for a second random sample of 150 customers were recorded. The mean waiting time was $\bar{x}$ minutes with a sample standard deviation of s minutes. A 99% confidence interval for μ is $17.78 < \mu < 19.22$. Find was $\bar{x}$ and s. (4 marks)

invNormCDf("C", 0.99, 1, 0)

−2.575829304 ✓✓

$$\left\{\begin{array}{l} x+2.575829304\times\frac{y}{\sqrt{150}}=19.22 \\ x-2.575829304\times\frac{y}{\sqrt{150}}=17.78 \end{array}\right|_{x,y}$$

{x=18.5, y=3.423426801}

$\bar{x} = 18.5$ minutes
$s = 3.42$ minutes ✓✓

(c) For $\sigma = 2.7$ determine the sample size n such that the 95% confidence interval for μ differs from the sample mean by no more than 0.5 minutes. (2 marks)

$$1.96\times\frac{2.7}{\sqrt{n}} \leq 0.5$$

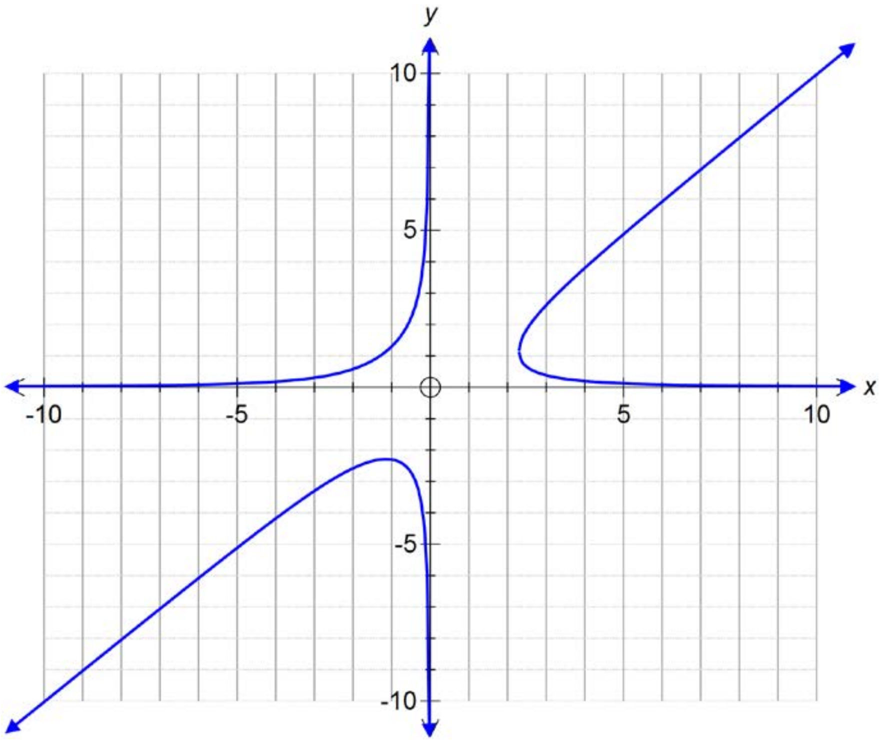
$$\left(\frac{1.96\times 2.7}{0.5}\right)^2 \leq n \quad ✓✓$$

$$112 \leq n$$

Question 9 **(12 marks)**

Consider the function with equation $x^2y - xy^2 = 3$

The graph of the function is shown below.



(a) Prove that $\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$ (4 marks)

$$x^2y + y^{.2}x - x^{.2}y.y' + y^2(-1) = 0$$ ✓✓

$$y'\left(x^2 - 2xy\right) = y^2 - 2xy$$ ✓

$$y' = \frac{y^2 - 2xy}{x^2 - 2xy}$$ as required. ✓

(b) Show that for relative extrema to exist, $y = 2x$. (4 marks)

For relative extrema, $\frac{dy}{dx} = 0$ ✓

$$\frac{y^2 - 2xy}{x^2 - 2xy} = 0$$

$$\therefore y^2 - 2xy = 0$$ ✓

$\therefore y = 0$ *or* $y = 2x$ but $y \neq 0$ (see graph) So $y = 2x$

✓✓

(c) Determine the exact co-ordinates of the point on the curve where the tangent is parallel to the *x* axis. (4 marks)

$x^2(2x) - x(4x^2) = 3$ ✓

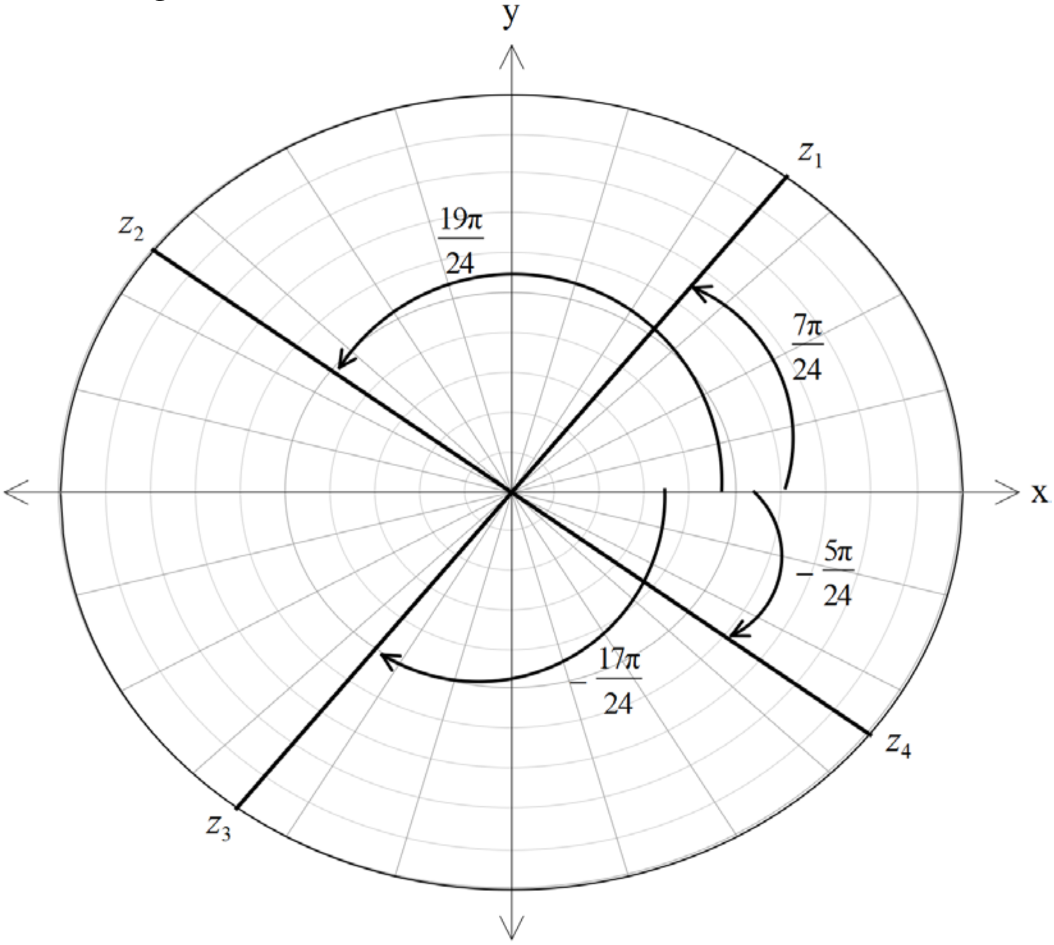
$x^3 = -\frac{3}{2}$ ✓

$x = -\left(\frac{3}{2}\right)^{\frac{1}{3}}$ and from (c) $y = -2\left(\frac{3}{2}\right)^{\frac{1}{3}}$ ✓✓

Question 10 **(5 marks)**

Find the roots of $\left(-2\sqrt{3}-2i\right)^{\frac{1}{4}}$ and locate them graphically on the diagram below.

Show some working.



✓ correctly locates all four roots

$$\tan\theta = \left(\frac{-2}{-2\cdot\sqrt{3}}\right) \Rightarrow \theta = \frac{\pi}{6} \Rightarrow \therefore \theta = \frac{7\pi}{6} \qquad r = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2} = 4$$

✓ finds ***θ*** and ***r*** values

$$z^4 = -2\sqrt{3}-2i = 4\ \text{CiS}\left(\frac{7\pi}{6}+2k\pi\right)$$

$$z = \left(-2\sqrt{3}-2i\right)^{\frac{1}{4}} = 4^{\frac{1}{4}}\ \text{CiS}\left(\frac{7\pi/6+2k\pi}{4}\right)$$

✓ represents in CiS form & takes fourth root

$$= 4^{\frac{1}{4}}\ \text{CiS}\left(\frac{7\pi}{24}+\frac{12k\pi}{24}\right)$$

if $k=0$, $z_0 = \sqrt{2}\ \text{CiS}(7\pi/24)$

if $k=1$, $z_1 = \sqrt{2}\ \text{CiS}(19\pi/24)$

if $k=2$, $z_2 = \sqrt{2}\ \text{CiS}(-17\pi/24)$

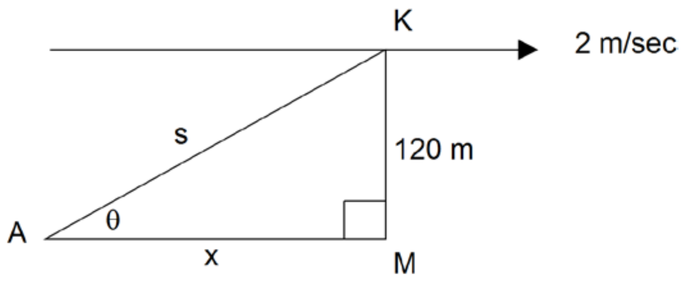
if $k=3$, $z_3 = \sqrt{2}\ \text{CiS}(-5\pi/24)$

✓✓ shows all 4 correct roots

NOTE: $-17\pi/24 = 31\pi/24$ and $-5\pi/24 = 43\pi/24$ // –1 if incorrect values stated

Question 11 **(7 marks)**

A kite flies parallel to the ground at a rate of 2 metres per second. A variable length of string, *s* metres, extends from a fixed point at A to the kite at K, and makes an angle of θ with the ground. The kite is a constant 120 metres above the ground. The diagram below shows the positions of A and K at some instant *t* seconds.



The position of M is such that $\tan\theta = \dfrac{120}{x}$.

(a) Prove that $\dfrac{d\theta}{dx} = -\dfrac{120}{s^2}$ (3 marks)

$$x = 120(\tan\theta)^{-1} \quad ✓$$

$$\frac{dx}{d\theta} = -120(\tan\theta)^{-2} \cdot \frac{1}{\cos^2\theta} = -\frac{120}{\sin^2\theta} \quad ✓$$

$$\therefore \frac{d\theta}{dx} = -\frac{\sin^2\theta}{120}$$

$$\text{But} \quad \sin\theta = \frac{120}{s} \quad \therefore \frac{d\theta}{dx} = -\frac{120}{s^2} \quad ✓$$

(b) At the instant when s = 240 metres, find the rate at which the angle θ is changing, in radians per second, and state whether the change is an increase or decrease.

(4 marks)

$$\frac{d\theta}{dx} = -\frac{120}{s^2}$$

$$\frac{d\theta}{dt} \cdot \frac{dt}{dx} = -\frac{120}{s^2} \quad ✓$$

$$\frac{d\theta}{dt} \cdot \frac{1}{2} = -\frac{120}{240^2} \quad ✓$$

$$\frac{d\theta}{dt} = -\frac{1}{240} \text{ radians /second} \quad ✓$$

i.e decrease of $\dfrac{1}{240}$ radians /second ✓

Question 12 **(10 marks)**

An object moving in a straight line undergoes simple harmonic motion with period π seconds and amplitude 8 cm. Initially the object is 4 cm to the right of the centre of oscillation and moving away from it.

(a) Determine, exactly, the initial velocity of the object. (5 marks)

$x = A\sin(nt + \alpha) = 8\sin(2t + \alpha)$ ✓

When t =0, x = 4 $\therefore \alpha = \frac{\pi}{6}$ ✓

$\therefore x = 8\sin\left(2t + \frac{\pi}{6}\right)$ ✓

$\therefore v = 16\cos\left(2t + \frac{\pi}{6}\right)$ ✓

$\therefore v_0 = 8\sqrt{3}$ cm/sec ✓

(b) State the maximum acceleration. (2 marks)

Max acceleration at extreme position ✓

$= -n^2 x = -4(-8) = 32$ cm/sec^2 ✓

(c) Calculate, correct to 2 decimal places, the total distance travelled in the first 3 seconds.

(3 marks)

$D = \int_0^3 \left| 16\cos\left(2t + \frac{\pi}{6}\right) \right| dt = 29.90$ cm ✓✓✓

Question 13 **(3 marks)**

Find the acute angle between the planes with equations $x+y-z=8$ and $2x-y+3z=-1$.

$x+y-z=8$ has a normal vector $\mathbf{n}_1=\begin{bmatrix}1 & 1 & -1\end{bmatrix}$ and
$2x-y+3z=-1$ has a normal vector $\mathbf{n}_2=\begin{bmatrix}2 & -1 & 3\end{bmatrix}$

If θ is the acute angle between the planes then

$$\theta=\cos^{-1}\left(\frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

✓ sets up the angle formula

$$=\cos^{-1}\left(\frac{|2+-1+-3|}{\sqrt{1+1+1}\times\sqrt{4+1+9}}\right)$$

$$=\cos^{-1}\frac{|-2|}{\sqrt{3}\times\sqrt{14}}=\cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$$

✓ substitutes values and simplfies

$\approx 72\cdot 02°$ *very* $\approx 73°$ or $1\cdot 26^R$

✓ correctly determines acute angle

Question 14 **(11 marks)**

Consider the complex number $z = cis\ \theta$

(a) Prove that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ (3 marks)

$z^n = \cos n\theta + i\sin n\theta$ ✓

$z^{-n} = \cos n\theta - i\sin n\theta$ ✓

By addition, $z^n + z^{-n} = 2\cos n\theta$ ✓

(b) Hence, show that $(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ (4 marks)

$(2\cos\theta)^6 = \left(z + z^{-1}\right)^6$ ✓

After expansion and collecting terms

$= \left(z^6 + z^{-6}\right) + 6\left(z^4 + z^{-4}\right) + 15\left(z^2 + z^{-2}\right) + 20$ ✓✓

$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ ✓

(c) Hence, determine an expression for $\cos^6 \theta$ in terms of cosines of multiple angles.

(1 mark)

$$\cos^6 \theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$ ✓

(d) Hence show that $\int_0^{\frac{\pi}{6}} \cos^6 \theta \, d\theta = \frac{5\pi}{96} + \frac{9\sqrt{3}}{64}$ (3 marks)

$$\int_0^{\frac{\pi}{6}} \cos^6 \theta \, d\theta$$

$$= \frac{1}{32}\left(\left[\frac{1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\,\theta\right]_0^{\frac{\pi}{6}}\right.$$ ✓✓

$$= \frac{1}{32}\left(\frac{9\sqrt{3}}{2} + \frac{5\pi}{3}\right)$$ ✓

Question 15 **(12 marks)**

A plane leaves an airport and climbs such that its position relative to the control tower is given by the vector $r = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 0.8 \end{pmatrix}$. The distance units are kilometres, and t is measured in minutes after take-off. The x and y axes are measured in the plane of the ground to the East and the North, respectively.

(a) Determine the position of the plane when it reaches its cruising height of 10 000 metres. (3 marks)

$10 = 0.8\,t$

$t = 12.5$ minutes. ✓

$$\textit{Position} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + 12.5\begin{pmatrix} 3 \\ 5 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 38.5 \\ 64.5 \\ 10 \end{pmatrix}$$ ✓✓

The coastline has equation $x + 2y = 70$.

(b) What is the height of the plane when it crosses the coastline? (3 marks)

Solve $x + 2y = 70$ and $1+3t = x$ and $2 + 5t = y$ ✓

$t = 5$ ✓

Height = (0.8)(5) = 4000 m ✓

(c) Calculate the speed of the plane relative to the ground. State your answer in kilometres per hour. (3 marks)

Leaves from $\begin{pmatrix}1\\2\\0\end{pmatrix}$ Crosses at $\begin{pmatrix}16\\27\\4\end{pmatrix}$ ✓

Direction relative to the ground is $\begin{pmatrix}15\\25\\0\end{pmatrix}$ ✓

Speed is $\left|\begin{pmatrix}15\\25\\0\end{pmatrix}\right| \div 5$ = 5.83 km/min = 353.13 km/hour ✓✓

(or use magnitude of velocity vector)

(d) Calculate the angle that the climbing plane makes with the horizontal. (3 marks)

$\begin{pmatrix}15\\25\\4\end{pmatrix} \cdot \begin{pmatrix}15\\25\\0\end{pmatrix} = \sqrt{15^2 + 25^2 + 4^2} . \sqrt{15^2 + 25^2 + 0^2} . \cos\theta$ ✓

$\cos\theta = 0.9907$ ✓

$\theta = 7.8^\circ$ ✓

Question 16 **(10 marks)**

The population of an island is currently 154 but its growth rate in the foreseeable future is expected to be given by $\frac{dP}{dt} = 0 \cdot 16P\left(1 - \frac{P}{500}\right)$ *ALGEBRAIC HINT:* $\frac{K}{x(K-x)} = \frac{1}{x} + \frac{1}{K-x}$

(a) Show that $\frac{P}{500-P} = k \cdot e^{0 \cdot 16t}$ [4]

$$\frac{dP}{dt} = 0 \cdot 16P\left(1 - \frac{P}{500}\right) = 0 \cdot 16\left(\frac{P(500-P)}{500}\right)$$

$$\therefore \quad \frac{500}{P(500-P)} \frac{dP}{dt} = 0 \cdot 16$$

$$\therefore \quad \int \frac{500}{P(500-P)} \frac{dP}{dt} dt = \int 0 \cdot 16 dt \quad \text{i.e.} \quad \int \frac{500}{P(500-P)} dP = \int 0 \cdot 16 dt$$

✓✓ rearranges the differential **and** separates the variables

$$\therefore \quad \int \left(\frac{1}{P} + \frac{1}{500-P}\right) dP = \int 0 \cdot 16 dt$$

$$\therefore \quad \ln|P| + \frac{1}{-1}\ln|500-P| = 0 \cdot 16t + C \quad \text{i.e.} \quad \ln\left|\frac{P}{500-P}\right| = 0 \cdot 16t + C$$

Hence $\frac{P}{500-P} = e^{0 \cdot 16t + C} = e^{C} \cdot e^{0 \cdot 16t} = k \cdot e^{0 \cdot 16t}$

✓ applies the algebraic hint and integrates correctly; ✓ simplifies the result

(b) Given that the population is currently 154, show that $P \approx \dfrac{500}{1+2\cdot 247e^{-0\cdot 16t}}$ [3]

$$\frac{P}{500-P} = k \cdot e^{0\cdot 16t} \quad \text{but when } t=0,\ P=154$$

$$\frac{154}{500-154} = k \quad \Rightarrow \quad k = 0\cdot 44509$$

✓ substitutes initial conditions to find ***k***

$$\therefore \quad \frac{P}{500-P} \approx 0\cdot 44509e^{0\cdot 16t}$$

$$\text{i.e.} \quad \frac{500-P}{P} \approx 2\cdot 247e^{-0\cdot 16t}$$

✓ reciprocates the fracion

$$\text{i.e.} \quad 500-P \approx 2\cdot 247Pe^{-0\cdot 16t}$$

$$\text{i.e.} \quad 500 \approx P\left(1+2\cdot 247e^{-0\cdot 16t}\right)$$

$$\therefore \ P \approx \frac{500}{1+2\cdot 247e^{-0\cdot 16t}}$$

✓ rearranges to make ***P*** the subject

continued overleaf / …

Question 19 *(continued)* **(11 marks)**

$$P \approx \frac{500}{1+2\cdot247e^{-0\cdot16t}}$$

(c) determine the population when ***t*** = 20 years [1]

$$P(20) \approx \frac{500}{1+2\cdot247e^{-0\cdot16\times20}} \approx 458 \text{ people}$$

✓ calculates the population

(d) find the time taken for the population to increase to 480. [1]

When ***P*** = 480

$$480 = \frac{500}{1+2\cdot247e^{-0\cdot16t}}$$

$$\therefore\ 1+2\cdot247e^{-0\cdot16t} = \frac{500}{480}$$

i.e. $2\cdot247e^{-0\cdot16t} \approx 0\cdot041667$

i.e. $e^{-0\cdot16t} \approx 0\cdot018543$

i.e. $-0\cdot16t \approx \ln(0\cdot018543)$

i.e. $t \approx \dfrac{\ln(0\cdot018543)}{-0\cdot16}$

$\therefore$ $t \approx 24\cdot9$ i.e. about 25 years

✓ calculates the time

(e) determine the limiting population size [1]

$$\text{as } t \to \infty,\ \ e^{-0\cdot16t} \to 0$$

$$\therefore\ P \to \frac{500}{1+0} = 500 \text{ people}$$

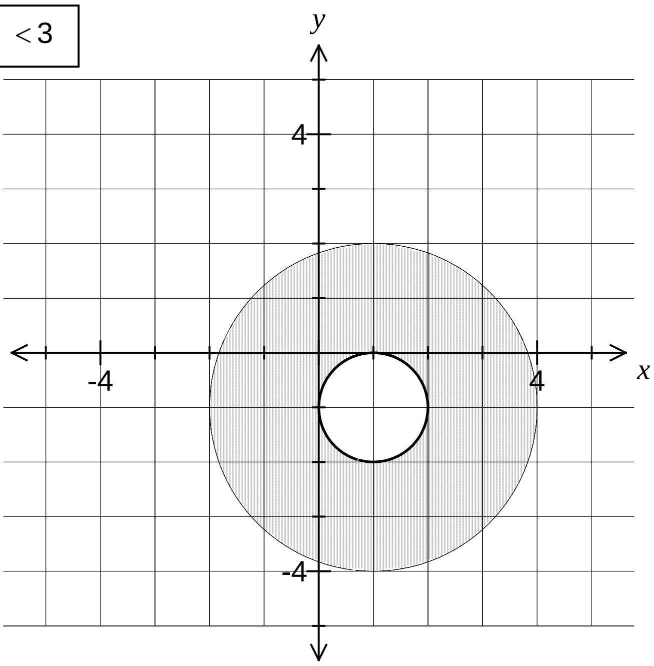
✓ calculates the limiting population size

Question 17 **(7 marks)**

Sketch the following regions in the complex plane.

(a) $1 \leq |z - 1 - i| < 3$. (4 marks)

$1 \leq |z - (1 + i)| < 3$



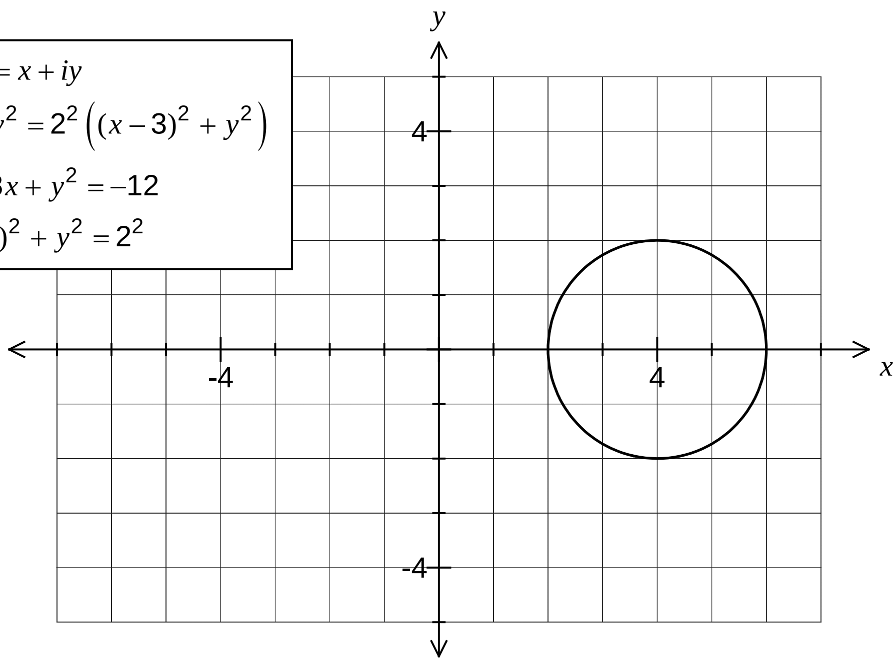
(b) $|z| = 2|z - 3|$. (3 marks)

Let $z = x + iy$

$x^2 + y^2 = 2^2\left((x-3)^2 + y^2\right)$

$x^2 - 8x + y^2 = -12$

$(x-4)^2 + y^2 = 2^2$

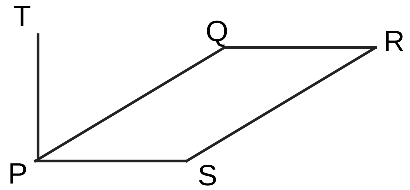


Question 18 **(13 marks)**

The points P, Q and R have position vectors $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{q} = 4\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ and $\mathbf{r} = 8\mathbf{i} + 21\mathbf{j} - 6\mathbf{k}$ respectively, relative to the origin. The point S has position vector $\mathbf{s}$ and is such that PQRS is a parallelogram.

(a) Find the position vector of $\mathbf{s}$ relative to the origin. (2 marks)

$$\mathbf{s} = \mathbf{p} + \mathbf{r} - \mathbf{q}$$
$$= 6\mathbf{i} + 17\mathbf{j} + \mathbf{k}$$



(b) Calculate the lengths of PQ and QR, the size of angle PQR and hence the area of the parallelogram. (4 marks)

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$$
$$= 2\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$$
$$|\overrightarrow{PQ}| = \sqrt{69}$$

$$\overrightarrow{QR} = \mathbf{r} - \mathbf{q}$$
$$= 4\mathbf{i} + 14\mathbf{j} - 2\mathbf{k}$$
$$|\overrightarrow{QR}| = 6\sqrt{6}$$

$$\angle PQR = 50.3° \text{ (using CAS)}$$

$$\text{Area} = 2 \times \frac{1}{2} \times \sqrt{69} \times 6\sqrt{6} \times \sin 50.3°$$
$$= 93.9 \text{ cm}^2$$

(c) Show that the vector $\mathbf{u} = 15\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ is perpendicular to the plane containing the parallelogram. (3 marks)

$$\mathbf{u} \bullet \overrightarrow{PQ} = \begin{bmatrix} 15 \\ -4 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 4 \\ -7 \end{bmatrix} = 30 - 16 - 14 = 0$$

$$\mathbf{u} \bullet \overrightarrow{QR} = \begin{bmatrix} 15 \\ -4 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -4 \\ -14 \\ 2 \end{bmatrix} = -60 + 56 + 4 = 0$$

As both dot products are zero and $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are non-parallel vectors in the plane, then **u** is perpendicular to the plane.

(d) The point T with position vector $\mathbf{t} = a\mathbf{i} + b\mathbf{j} + 4\mathbf{k}$ lies on the line that is perpendicular to the plane, through P. Determine the volume of the pyramid PQRST. (4 marks)

T lies on the line $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 15 \\ -4 \\ 2 \end{bmatrix}$

Using **k** coefficient: $3 + 2\lambda = 4 \quad \Rightarrow \lambda = 0.5$

$$\mathbf{t} = 9.5\mathbf{i} + \mathbf{j} + 4\mathbf{k}$$

$$\overrightarrow{PT} = 7.5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$|\mathbf{t}| = \frac{7\sqrt{5}}{2} \quad (\approx 7.826)$$

Volume of pyramid $= \frac{1}{3} \times A \times h$

$= \frac{1}{3} \times 93.91 \times 7.826$

$= 245 \text{ cm}^3$

END OF QUESTIONS

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